New solutions to the reflection equation and the projecting method

1999 J. Phys. A: Math. Gen. 32 1547

(http://iopscience.iop.org/0305-4470/32/9/005)

J. Phys. A: Math. Gen. **32** (1999) 1547–1555. Printed in the UK PII: S0305-4470(99)98744-9

New solutions to the reflection equation and the projecting method

Holger Frahm† and Nikita A Slavnov‡

† Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany
‡ Steklov Mathematical Institute, Moscow 117966, Russia

Received 26 October 1998, in final form 7 December 1998

Abstract. New integrable boundary conditions for integrable quantum systems can be constructed by tuning of scattering phases due to reflection at a boundary and an adjacent impurity and subsequent projection onto subspaces. We illustrate this mechanism by considering a $gl(m < n)$-impurity attached to an open $gl(n)$-invariant quantum chain and a Kondo spin S coupled to the supersymmetric t–J model.

1. Introduction

Studies of integrable models with open boundary conditions have attracted much interest recently. The exact solutions of these systems provide important insights into the nature of bound states due to the presence of local potentials and properties of impurities coupled to one-dimensional quantum systems [1–9].

The classification of open boundary conditions for integrable quantum chains is possible within the framework of the quantum inverse scattering method (QISM) [10] by supplementing the Yang–Baxter equation—which guarantees the factorizability of N-particle scattering processes in the bulk of the system—with the reflection equation (RE) algebra to ensure compatibility of two-particle scattering and particle–boundary scattering [11, 12]. The simplest solutions to this RE algebra are c-number matrices with entries corresponding to the phase shifts due to (static) boundary fields in the different channels. In general, such boundary fields will break the symmetry of the model: in spin chains they have been identified as magnetic fields acting on the boundary sites [12], for the $gl(2|1)$-invariant (supersymmetric) t–J model the (diagonal) c-number solutions of the RE correspond to boundary chemical potential and boundary magnetic fields, respectively [2, 13]. Dynamic impurities located at the boundary can also be described in terms of solutions to the RE: as observed in [12] 'dressing' of c-number boundary matrices with local monodromy matrices generates new solutions to the RE with elements acting non-trivially in an impurity Hilbert space. Such operator-valued solutions to the RE—called 'regular' in the following—have been used to construct models of spin-S chains with spin-S' impurities located on the boundary site (see, e.g., [4, 5]). All of these models are similar in that operators acting on the quantum space of the impurity need to be chosen among representations of the *same* algebra as the ones acting on the bulk sites, for example $SU(2)$ for Heisenberg models or $gl(2|1)$ for the supersymmetric t–J model, just as in the corresponding closed chain systems [14].

Integrable models of Kondo impurities in one-dimensional electronic continuum [6, 7] (recently rediscovered in [15]) and lattice models [8, 9] which have been solved by means of the coordinate Bethe ansatz appear not to fit into this scheme: in these systems the quantum space of the impurity is a projection of the symmetry group onto a subgroup acting only on the spin-degree of freedom. Recently, Zhou and coworkers [16, 17] have succeeded in formulating the model of a Kondo impurity in the $gl(2|1)$-symmetric t–J model [8, 9] in the framework of the RE algebra. They have found an operator-valued solution to the RE which apparently cannot be obtained by the 'regular' dressing procedure with $gl(2|1)$-symmetric monodromy matrices containing the impurity degrees of freedom. Instead, they propose a decomposition into 'singular' matrices with $SU(2)$ spin operators as entries.

In this paper, we introduce a method which allows projection of 'regular' solutions of the RE to a certain subspace of the impurity's Hilbert space after adjusting the boundary phase shifts of the c-number matrix to the ones due to the dressing impurity. In the following section we briefly review the RE formalism and formulate the necessary conditions for the application of the projection method. In section 3 we apply this method to the case of $gl(n)$ algebra. Finally, we show how to obtain the 'singular' boundary matrices of [16, 17] within this approach.

2. General method

Before consideration of the specific cases we would like to formulate our approach in general.

The classification of integrable boundary conditions within the QISM is based on representations of two algebras $\mathcal{T}_\pm$ [12]. The RE for $\mathcal{T}_-(u)$ has the form

$$R_{12}(u_1-u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1+u_2)\overset{2}{\mathcal{T}}_-(u_2)=\overset{2}{\mathcal{T}}_-(u_2)R_{12}(u_1+u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1-u_2). \quad (2.1)$$

Here we use standard notations $\overset{1}{\mathcal{T}}_-(u)=\mathcal{T}_-(u)\otimes I$ and $\overset{2}{\mathcal{T}}_-(u)=I\otimes\mathcal{T}_-(u)$. The RE for $\mathcal{T}_+(u)$ will not be considered in the present paper: the solutions of these equations are related to (2.1) by an automorphism [12]. In the Hamiltonian limit $\mathcal{T}_\pm(u)$ determine the boundary terms of the quantum chain. For example, the solutions of (2.1) lead to an operator $\partial_u\overset{1}{\mathcal{T}}_-(u=0)$ acting on the first site of the chain. For details we refer the reader to [12].

The R-matrix satisfies the quantum Yang–Baxter equation (YBE)

$$R_{12}(u)R_{13}(u+v)R_{23}(v)=R_{23}(v)R_{13}(u+v)R_{12}(u). \quad (2.2)$$

As usual $R_{21}(u)=P_{12}R_{12}(u)P_{12}$, where P_{12} is the permutation operator. The unitarity property of the R-matrix is assumed to hold

$$R_{12}(u)R_{21}(-u)=\rho(u) \quad (2.3)$$

where $\rho(u)$ is a scalar function.

As we have mentioned already in the introduction, operator-valued (quantum) solutions of the RE (2.6) can be constructed following [12]: let $L(u)$ be a quantum solution of the intertwining equation of the QISM:

$$R_{12}(u_1-u_2)\overset{1}{L}(u_1)\overset{2}{L}(u_2)=\overset{2}{L}(u_2)\overset{1}{L}(u_1)R_{12}(u_1-u_2). \quad (2.4)$$

The entries of the L-operator are quantum operators, acting in a Hilbert space $\mathcal{H}$.

Given a solution of (2.4) we define an operator-valued matrix $K_-(u)$ as

$$K_-(u)=L(u)\mathcal{T}(u)L^{-1}(-u) \quad (2.5)$$

where $\mathcal{T}(u)$ is a c-number solution of (2.1). Then one can check [12] that the quantum $K_-(u)$ boundary matrix solves the RE:

$$R_{12}(u_1-u_2)\overset{1}{K}_-(u_1)R_{21}(u_1+u_2)\overset{2}{K}_-(u_2)=\overset{2}{K}_-(u_2)R_{12}(u_1+u_2)\overset{1}{K}_-(u_1)R_{21}(u_1-u_2). \quad (2.6)$$

In what follows we shall refer to the formula (2.5) as 'regular' factorization. Similarly, we call the corresponding K-matrix the 'regular' solution of the RE.

In [17] a new type of RE solution was found. This new K-matrix cannot be presented in the form (2.5). Instead, the authors proposed so-called 'singular' factorization

$$K_-(u) \equiv K_s(u) = \lim_{\epsilon \to 0} L_\epsilon(u) L_\epsilon^{-1}(-u) \tag{2.7}$$

where the L_ϵ-operator depends on the auxiliary parameter ϵ. The special feature of this solution is that factorization (2.7) is valid for arbitrary ϵ (i.e. $K_s(u)$ does not depend on ϵ) which allows one to omit the limit in (2.7). On the other hand, the operator L_ϵ satisfies the intertwining relation (2.4) in the limit $\epsilon \to 0$ only, but the limit $\epsilon \to 0$ for $L_\epsilon^{-1}(u)$ does not exist. Following the authors of [17] we call the representation (2.7) 'singular' factorization and the corresponding K-matrix the 'singular' solution of the RE, in spite of its well-defined limit for $\epsilon \to 0$.

In the present paper we show that these 'singular' solutions are nothing but projections of suitably chosen 'regular' ones. Our approach is based on the following simple observation. Consider some 'regular' solution of the RE, obtained by the standard procedure (2.5). The entries of such a quantum K-matrix are operators, acting in the same space $\mathcal{H}$ as the entries of the L-operator. Now consider two orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, such that $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$, characterized by projectors π_1 and π_2, respectively. Then it is easily seen that vanishing of one of the projections $\pi_1 K_-(u)\pi_2$ or $\pi_2 K_-(u)\pi_1$,

$$\pi_1 K_-(u)\pi_2 = 0 \qquad \text{or} \qquad \pi_2 K_-(u)\pi_1 = 0 \tag{2.8}$$

implies that the projections $\pi_1 K_-(u)\pi_1$ and $\pi_2 K_-(u)\pi_2$ of the operator $K_-(u)$ onto the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ solve the RE:

$$\begin{aligned} R_{12}(u_1 - u_2)(\pi_i \overset{1}{K_-}(u_1)\pi_i) R_{21}(u_1 + u_2)(\pi_i \overset{2}{K_-}(u_2)\pi_i) \\ = (\pi_i \overset{2}{K_-}(u_2)\pi_i) R_{12}(u_1 + u_2)(\pi_i \overset{1}{K_-}(u_1)\pi_i) R_{21}(u_1 - u_2) \end{aligned} \tag{2.9}$$

where $i = 1, 2$. Thus, new quantum solutions of the RE can be generated via projection of the original K-matrix onto a subspace of its quantum Hilbert space.

The first problem, however, is to find the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$, possessing the property (2.8). For an arbitrary $K_-(u)$ boundary matrix such a decomposition may not exist. Nevertheless, as will be demonstrated later, this decomposition is possible for certain solutions of the RE of the type (2.5) where the c-number factor has been properly adjusted to the dressing L-operators. In particular, the solution of the RE found in [17] can be obtained by the method described above.

The second problem related to this method is whether the projection provides us with *really* new solutions of the RE, i.e. ones not allowing 'regular' factorization. It is easy to see that this is not always so. If, for example, $\mathcal{H}_2$ is a one-dimensional subspace, then evidently the projection $\pi_2 K_-(u)\pi_2$ is just one of the known c-number solutions of the RE.

Apart from this trivial possibility, the examples considered in the following do not permit the formulation of a criterion which would allow one to predict that a projection of a 'regular' solution is not 'regular'. However, we shall demonstrate that 'singular' solutions can be obtained via the projection procedure.

3. The case of $gl(n)$ algebra

In this section we demonstrate the method of projection, using the example of $gl(n)$ algebra. Consider an $n^2 \times n^2$ R-matrix

$$R(u) = uI + P \tag{3.1}$$

where the permutation operator P has the entries $P^{\alpha\beta}_{jk} = \delta_{j\beta}\delta_{k\alpha}$. The simplest quantum L-operator satisfying equation (2.4) has the form

$$L_{ij}(u) = \frac{1}{u+1}(\delta_{ij}u + |j\rangle\langle i|). \tag{3.2}$$

Here

$$\langle i| = (\underbrace{0,\ldots,0}_{i-1}, 1, 0\ldots, 0) \qquad |i\rangle = (\langle i|)^{\mathrm{T}}. \tag{3.3}$$

In fact, this L-operator coincides with the R-matrix (3.1) up to a normalization factor. The entries of the L-operator act in the quantum space $\mathcal{H} = C^n$.

Let us introduce two quantum projectors π_1 and π_2

$$\pi_1 = \sum_{k=1}^{m}|k\rangle\langle k| \qquad \pi_2 = \sum_{k=m+1}^{n}|k\rangle\langle k| \qquad \pi_1 + \pi_2 = I_q \tag{3.4}$$

where m is a fixed number from the interval $1 \leqslant m \leqslant n$, and I_q is the identity operator in $\mathcal{H}$. Obviously, these projectors define two orthogonal subspaces: $\mathcal{H}_1 = \mathrm{span}\{|1\rangle,\ldots,|m\rangle\}$ and $\mathcal{H}_2 = \mathrm{span}\{|m+1\rangle,\ldots,|n\rangle\}$. As a first stage we are going to construct a 'regular' K-matrix by means of the L-operator (3.2) and some c-number solution of the RE. Then we shall consider the projections of this K-matrix onto subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

We start with the solution of the RE

$$K_-(u) = L(u+c)\mathcal{T}(u)L^{-1}(-u+c). \tag{3.5}$$

Here c is a constant and $\mathcal{T}(u)$ is a diagonal c-number solution of the RE breaking the $gl(n)$-symmetry of the system down to $gl(m)$ [18]:

$$\mathcal{T}_{ij}(u) = \delta_{ij}h_i(u). \tag{3.6}$$

Here

$$h_i(u) = 1 \quad \text{for } i \leqslant m \qquad h_i(u) \equiv h(u) = \frac{\xi - u}{\xi + u} \quad \text{for } i > m \tag{3.7}$$

with some constant ξ.

With the normalization in (3.2) we have $L^{-1}(-u) = L(u)$. Thus we arrive at $K_-(u) = K_d(u) + K_a(u)$, where

$$\begin{aligned}(K_d(u))_{ij} &= \frac{\delta_{ij}}{(u+1)^2 - c^2}\left[(u^2 - c^2)h_i(u) + \sum_{k=1}^{n}h_k(u)|k\rangle\langle k|\right]\\ (K_a(u))_{ij} &= \frac{1}{(u+1)^2 - c^2}[(u+c)h_i(u) + (u-c)h_j(u)]|j\rangle\langle i|.\end{aligned} \tag{3.8}$$

Thus, the 'regular' solution of the RE (2.6) is constructed. Next let us consider the projections of this solution. First, we have to adjust the parameters in (3.8) such that $\pi_1 K_-(u)\pi_2 = 0$ or $\pi_2 K_-(u)\pi_1 = 0$. The projections of the part $K_d(u)$ are automatically equal to zero

$$\pi_1 K_d(u)\pi_2 = \pi_2 K_d(u)\pi_1 = 0. \tag{3.9}$$

As for the projections of the part $K_a(u)$, we have

$$(\pi_1 K_a(u)\pi_2)_{ij} = \begin{cases}(K_a(u))_{ij} & i > m,\ j \leqslant m\\ 0 & \text{otherwise}\end{cases} \tag{3.10}$$

$$(\pi_2 K_a(u)\pi_1)_{ij} = \begin{cases}(K_a(u))_{ij} & i \leqslant m,\ j > m\\ 0 & \text{otherwise.}\end{cases} \tag{3.11}$$

Thus, by choosing $\xi = \pm c$ in (3.7) we obtain $\pi_1 K_-(u)\pi_2 = 0$ ($\pi_2 K_-(u)\pi_1 = 0$). In either case the projections $\pi_1 K_-(u)\pi_1$ and $\pi_2 K_-(u)\pi_2$ satisfy the RE. We would like to emphasize, in particular, that the parameter ξ in the c-number solution (3.6) has to be adjusted to the parameter c in the dressing L-operators for the projections $\pi_1 K_-(u)\pi_2$ and $\pi_2 K_-(u)\pi_1$ to vanish.

Let us now focus on $\xi = c$: in this case the projected reflection matrices are

$$
\begin{aligned}
(\pi_1 K_-(u)\pi_1)_{ij} &= \begin{cases} \dfrac{(u^2 - c^2 + 1)\delta_{ij} + 2u|j\rangle\langle i|}{(u+1)^2 - c^2} & i, j \leqslant m \\ \delta_{ij}\dfrac{c+1-u}{c+1+u} & \text{otherwise} \end{cases} \\
(\pi_2 K_-(u)\pi_2)_{ij} &= \frac{c-u}{c+u}\begin{cases} \dfrac{(u^2 - c^2 + 1)\delta_{ij} + 2u|j\rangle\langle i|}{(u+1)^2 - c^2} & i, j > m \\ \delta_{ij}\dfrac{c-1+u}{c-1-u} & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.12}
$$

Introducing L-operators, acting in the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ only,

$$
\begin{aligned}
(L_1(u))_{ij} &= (u+c)\delta_{ij} + |j\rangle\langle i| \qquad i, j \leqslant m \\
(L_2(u))_{ij} &= (u-c)\delta_{ij} + |j\rangle\langle i| \qquad i, j > m
\end{aligned} \tag{3.13}
$$

the projections (3.12) can be presented as block matrices

$$
\begin{aligned}
\pi_1 K_-(u)\pi_1 &= \frac{c+1-u}{c+1+u}\begin{pmatrix} L_1(u)L_1^{-1}(-u) & 0 \\ 0 & 1 \end{pmatrix} \\
\pi_2 K_-(u)\pi_2 &= \frac{c-u}{c+u}\,\frac{c-1+u}{c-1-u}\begin{pmatrix} 1 & 0 \\ 0 & L_2(u)L_2^{-1}(-u) \end{pmatrix}.
\end{aligned} \tag{3.14}
$$

Clearly, the external factors can be removed, and we arrive at two new solutions of the RE

$$
K_{s1}(u) = \begin{pmatrix} L_1(u)L_1^{-1}(-u) & 0 \\ 0 & 1 \end{pmatrix} \qquad K_{s2}(u) = \begin{pmatrix} 1 & 0 \\ 0 & L_2(u)L_2^{-1}(-u) \end{pmatrix}. \tag{3.15}
$$

While these solutions cannot be presented as *regular* solutions (2.5) of the RE they can be factorized in terms of *singular* solutions to (2.4): with

$$
L_\epsilon(u) = \begin{pmatrix} L_1(u) & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{3.16}
$$

we can write $K_{s1}(u) = L_\epsilon(u)L_\epsilon(-u)^{-1}$. However, the operator $L_\epsilon(u)$ satisfies equation (2.4) only in the limit $\epsilon \to 0$. Thus, we have a complete analogy with the case considered in [17].

In conclusion of this section we would like to mention some properties of 'singular' factorization, which make it essentially different from the 'regular' one. First, inserting a c-number solution $\mathcal{T}(u)$ between dressing L_ϵ-operators

$$
K_{s,\mathcal{T}} = L_\epsilon(u)\mathcal{T}(u)L_\epsilon^{-1}(-u) \tag{3.17}
$$

we do not arrive at a new RE solution. The matrix (3.17) does not satisfy the RE. One should not be surprised at this fact, since, as we have seen, vanishing of projections $\pi_1 K_-(u)\pi_2$ (or $\pi_2 K_-(u)\pi_1$) was provided only due to the special choice of the $\mathcal{T}$-matrix (3.6).

Second, in the 'regular' case one can generate new K-matrices via the replacement

$$
L(u) \to T(u) = L_N(u)\ldots L_1(u)
$$

where $L_i(u)$ are copies of the original L-operator, acting in different quantum spaces. For the 'singular' factors (3.16) this method fails, i.e. if $T_\epsilon = L_{\epsilon,N}\ldots L_{\epsilon,1}$, then

$K_{s,T}(u) = T_\epsilon(u)T_\epsilon^{-1}(-u)$ does not solve the RE. This fact can also be explained in the framework of the projecting method. The problem is that the subtle tuning of the boundary and impurity properties which leads to the fulfilment of the necessary condition (2.8) cannot be performed in the large quantum space $\mathcal{H}_T$ of the matrices $T_\epsilon(u)$ and $K_{s,T}(u)$. This makes it impossible to find a decomposition $\mathcal{H}_T = \mathcal{H}_{T,1} \oplus \mathcal{H}_{T,2}$.

4. Kondo impurity in the supersymmetric *t–J* model

Our second example deals with the Kondo impurity in the supersymmetric t–J model recently constructed in [8, 9, 16, 17]. Integrability of the periodic model is proven by construction of the enveloping vertex model within a $\mathbf{Z}_2$-graded extension of the QISM [19–21]. A similar extension of the RE is necessary, for the algebra $\mathcal{T}_-$ it is formally identical to the ungraded case (2.6) with a 9×9 R-matrix

$$R_{12}(u) = uI + P_{12}. \tag{4.1}$$

Here P_{12} is the $\mathbf{Z}_2$-graded permutation operator

$$(P_{12})^{\alpha\beta}_{jk} = (-1)^{[j][\alpha]}\delta_{j\beta}\delta_{k\alpha}. \tag{4.2}$$

The $\mathbf{Z}_2$-grading is chosen in such a way that $[1] = [2] = 1$ and $[3] = 0$. The R-matrix (4.1) satisfies the unitarity property (2.3).

The diagonal c-number solutions of the RE are again of the form (3.6) and correspond to boundary magnetic fields and chemical potentials, respectively [2, 13]. Recently, a new type of quantum solution of the RE (2.6) has been found [17]:

$$K_s(u) = \begin{pmatrix} \alpha(u) + \beta(u)S^z & \beta(u)S^- & 0 \\ \beta(u)S^+ & \alpha(u) - \beta(u)S^z & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Here S^z and $S^\pm$ are the usual generators of a $SU(2)$ algebra: $[S^z, S^\pm] = \pm S^\pm$, $[S^+, S^-] = 2S^z$, $\mathbf{S}^2 = s(s+1)$. The functions $\alpha(u)$ and $\beta(u)$ are equal to

$$\begin{aligned} \alpha(u) &= \frac{(c+s+1/2)(c-s-1/2) - u^2 + u}{(c+s+1/2-u)(c-s-1/2-u)} \\ \beta(u) &= \frac{2u}{(c+s+1/2-u)(c-s-1/2-u)} \end{aligned} \tag{4.4}$$

with a constant c.

The general structure of the K-matrix (4.3) looks very similar to (3.12) and (3.15). Indeed, this solution can be presented in terms of ‘singular’ factorization [17]

$$K_s(u) = L_\epsilon(u){L_\epsilon}^{-1}(-u) \tag{4.5}$$

where

$$L_\epsilon(u) = \begin{pmatrix} u - c - 1 - S^z & -S^- & 0 \\ -S^+ & u - c - 1 + S^z & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{4.6}$$

Just as in our previous example, the operator L_ϵ satisfies the (graded version of the) intertwining equation (2.4) in the limit $\epsilon \to 0$ only. All the ‘pathological’ properties of the ‘singular’ solutions, listed at the end of the previous section, are valid for the K-matrix (4.3). This leads us to assume that in fact the K-matrix (4.3) is nothing but a projection of a ‘regular’ solution of the RE.

To reproduce the result (4.3) of [17] by means of the projection method we have to consider solutions of the intertwining relation (2.4) and the reflection equation (2.6), which are invariant under the action of the graded Lie algebra $gl(2|1)$ (see, e.g., [22, 23]). Apart from the generators 1, S^z and $S^\pm$ which form an (ungraded) $gl(2)$ subalgebra, there is an additional generator B of even parity (charge), commuting with the spin operators, and four odd generators $V^\pm$ and $W^\pm$. The commutation relations between even and odd generators are listed as follows:

$$\begin{aligned} &[S^z, V^\pm] = \pm\tfrac{1}{2}V^\pm \qquad [S^\pm, V^\pm] = 0 \qquad [S^\mp, V^\pm] = V^\mp \\ &[S^z, W^\pm] = \pm\tfrac{1}{2}W^\pm \qquad [S^\pm, W^\pm] = 0 \qquad [S^\mp, W^\pm] = W^\mp \\ &[B, V_\pm] = \tfrac{1}{2}V_\pm \qquad [B, W_\pm] = -\tfrac{1}{2}W_\pm. \end{aligned} \tag{4.7}$$

The odd generators satisfy anticommutation relations

$$\begin{aligned} &\{V^\pm, V^\pm\} = \{V^\pm, V^\mp\} = \{W^\pm, W^\pm\} = \{V^\pm, W^\mp\} = 0 \\ &\{V^\pm, W^\pm\} = \pm\tfrac{1}{2}S^\pm \qquad \{V^\pm, W^\mp\} = \tfrac{1}{2}(S^z \pm B). \end{aligned} \tag{4.8}$$

In the following we shall consider the 'atypical' representation $[s]_+$ of this algebra [22, 23]. In a basis $\{|b, s, m\rangle\}$ where B, $\mathbf{S}^2$ and S^z are diagonal, this representation contains two spin multiplets of spin s and $s - 1/2$ with charge $b = s$ and $s + 1/2$, respectively:

$$\mathcal{H}_1 = \mathrm{span}\{|s, s, m\rangle\} \qquad \mathcal{H}_2 = \mathrm{span}\{|s + 1/2, s - 1/2, m\rangle\}. \tag{4.9}$$

The non-vanishing matrix elements of the remaining operators are

$$\begin{aligned} &\left\langle s + \frac{1}{2}, s - \frac{1}{2}, m \pm \frac{1}{2}\middle| S^\pm \middle| s + \frac{1}{2}, s - \frac{1}{2}, m \mp \frac{1}{2}\right\rangle = \sqrt{s^2 - m^2} \\ &\left\langle s + \frac{1}{2}, s - \frac{1}{2}, m \pm \frac{1}{2}\middle| V^\pm \middle| s, s, m\right\rangle = \pm\sqrt{\frac{s \mp m}{2}} \\ &\left\langle s, s, m\middle| W^\pm \middle| s + \frac{1}{2}, s - \frac{1}{2}, m \mp \frac{1}{2}\right\rangle = \sqrt{s \pm m2}. \end{aligned} \tag{4.10}$$

Now we consider the following 'regular' quantum solution of the RE

$$K_-(u) = L(u + c)\mathcal{T}(u)L^{-1}(-u + c). \tag{4.11}$$

Here $\mathcal{T}(u)$ is the c-number solution of the RE corresponding to a boundary chemical potential

$$\mathcal{T}(u) = \mathrm{diag}\left(1, 1, \frac{\xi - u}{\xi + u}\right) \tag{4.12}$$

and the L-operator containing the degrees of freedom of the quantum impurity in (4.11) is equal to [19]

$$(u - s - 1/2)L(u) = u - s - 1/2 + \begin{pmatrix} B - S^z & -S^- & -\sqrt{2}V^- \\ -S^+ & B + S^z & \sqrt{2}V^+ \\ \sqrt{2}W^+ & \sqrt{2}W^- & 2B \end{pmatrix}. \tag{4.13}$$

We have chosen the normalization such that $L^{-1}(-u + c) = L(u - c)$.

For the projection of the 'regular' K-matrix (4.11) we use the decomposition of the impurity quantum space $\mathcal{H}$ into the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of spaces (4.9). One can find the projection of (4.11) onto subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ by computing the projections of the L-operator and using

$$\pi_i K_- \pi_j = [\pi_i L \pi_1]\mathcal{T}[\pi_1 L^{-1} \pi_j] + [\pi_i L \pi_2]\mathcal{T}[\pi_2 L^{-1} \pi_j] \tag{4.14}$$

where π_i are projectors onto $\mathcal{H}_i$, as before. These calculations are quite straightforward, therefore we summarize the results only. The condition $\pi_1 K_-(u)\pi_2 = 0$ is satisfied by choosing $\xi = c + s - 1/2$ in (4.12). Then the projection $\pi_1 K_-(u)\pi_1$ exactly coincides with the matrix $K_s(u)$ (4.3). Thus, as we have stated previously, the 'singular' RE solution of [17] is indeed the projection of the 'regular' solution (4.11).

5. Conclusion

We have presented a method which allows—by adjusting the parameters of the c-number boundary matrix and those of an adjacent dynamical impurity—the construction of new quantum solutions of the RE by means of the projection (2.9). Since $K_-(u)$ is directly related to the boundary term of the corresponding quantum Hamiltonian [12], satisfying condition (2.8) amounts to (block-) diagonalization of the Hamiltonian in the Hilbert space of the impurity. While each of these blocks may correspond to a previously known boundary condition—as trivially seen when projecting to a one-dimensional subspace—we have presented several cases where new representations of the RE algebra arise which do not allow the presentation in terms of 'regular' factorization (2.5). These new cases include models for a $gl(m < n)$-spin impurity coupled to a $gl(n)$-symmetric quantum chain and the case of an $SU(2)$ Kondo spin in the supersymmetric t–J chain [16, 17]. A common feature of these 'singular' solutions to the RE is a remaining non-trivial symmetry in the impurity degrees of freedom *after* projection.

Note that the applicability of the projection method introduced in this paper is *not* restricted to the case of models with rational R-matrices considered in the previous examples: in fact, the generalization of the results of section 3 to systems with trigonometric R-matrices and corresponding c-number boundary matrices is straightforward.

The existence of projected boundary matrices has important consequences for the solution of systems with open boundary conditions by means of the algebraic Bethe ansatz: proper choice of a suitable reference state, which needs to be contained in the projected Hilbert space, is crucial to capture the properties of the impurity site. This statement holds, in particular, for the graded models, such as the t–J model where *different* Bethe ansätze are possible starting from various fully polarized states.

Finally, we would like to emphasize the remark of [17] regarding Kondo impurities in *closed* chains: it is obvious from the discussion above that the presence of a boundary next to the quantum impurity is essential for our construction. Using a 'singular' L-operator such as $L_{\epsilon\to 0}(u)$ from (4.6) to construct a periodic chain leads to the Heisenberg model with the impurity of Andrei and Johannesson [14] rather than a Kondo spin in a t–J model.

Acknowledgment

This work has been supported by the Deutsche Forschungsgemeinschaft under grant No Fr 737/2–3.

References

[1] Kapustin A and Skorik S 1996 *J. Phys. A: Math. Gen.* **29** 1629
[2] Eßler F H L 1996 *J. Phys. A: Math. Gen.* **29** 6183
[3] Bedürftig G and Frahm H 1997 *J. Phys. A: Math. Gen.* **30** 4139
(Bedürftig G and Frahm H 1997 *Preprint* cond-mat/9702227)
[4] Frahm H and Zvyagin A A 1997 *J. Phys.: Condens. Matter* **9** 9939
[5] Wang Y 1997 *Phys. Rev.* B **56** 14 045
[6] Schulz H 1986 *J. Phys. C: Solid State Phys.* **20** 2375
[7] Göhmann F and Schulz H 1990 *J. Phys.: Condens. Matter* **2** 3841
[8] Wang Y, Dai J-H, Hu Z-N and Pu F-C 1997 *Phys. Rev. Lett.* **79** 1901
(Wang Y, Dai J-H, Hu Z-N and Pu F-C 1997 *Preprint* cond-mat/9706086)
[9] Hu Z-N, Pu F-C and Wang Y 1998 *J. Phys. A: Math. Gen.* **31** 5241
(Hu Z-N, Pu F-C and Wang Y 1998 *Preprint* cond-mat/9804032)
[10] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)

[11] Cherednik I V 1984 *Theor. Math. Phys.* **61** 977
[12] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[13] González-Ruiz A 1994 *Nucl. Phys.* B **424** 468
[14] Andrei N and Johannesson H 1984 *Phys. Lett.* A **100** 108
[15] Wang Y and Voit J 1996 *Phys. Rev. Lett.* **77** 4934
[16] Zhou H-Q and Gould M D 1998 *Preprint* cond-mat/9809055
[17] Zhou H-Q, Ge X-Y, Links J and Gould M D 1998 *Preprint* cond-mat/9809056
[18] de Vega H J and González-Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L519
(de Vega H J and González-Ruiz A 1992 *Preprint* hep-th/9211114)
[19] Kulish P P 1986 *J. Sov. Math.* **35** 2648
[20] Eßler F H L and Korepin V E 1992 *Phys. Rev.* B **46** 9147
[21] Foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611
[22] Scheunert M, Nahm W and Rittenberg V 1977 *J. Math. Phys.* **18** 155
[23] Marcu M 1980 *J. Math. Phys.* **21** 1277